A Bias Correction Method for Realized Covariance Calculated Using Previous Tick Interpolation

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January 14, 2006, Yokohama

Introduction

■ Estimation of cross-volatility (Conditional covariance of asset return).
$\Rightarrow$ Portfolio risk, etc.

■ High-frequency data.
e.g. Hourly data, 30 minutes data, $\cdots$, 5 minutes data, Transaction data (Raw data)

$\Rightarrow$ **Realized Covariance**

DGP:

$$\underbrace{dp\,(t)}_{n\times 1} = \underbrace{\Sigma\,(t)}_{n\times n}\underbrace{dz\,(t)}_{n\times 1}, \quad 0 \le t \le T$$

(Instantaneous or spot) volatility matrix:

$$\Omega(t) \equiv \Sigma\,(t)\,\Sigma\,(t)'$$

$\boxed{\text{Estimation of integrated volatility } \int_0^T \Omega(t)dt}$

$\Rightarrow$ Quadratic variation:

$$\lim_{M\to\infty} \underbrace{\sum_{m=1}^{M} \Delta p(mT/M)\Delta p(mT/M)'}_{\text{Realized volatility}} = \int_0^T \Omega(t)dt$$

Intuitively,

$$\int_0^T dp(t)dp(t)' = \int_0^T \Sigma(t) \underbrace{dz(t)dz(t)'}_{dtI_n} \Sigma(t)'$$

Discrete observation

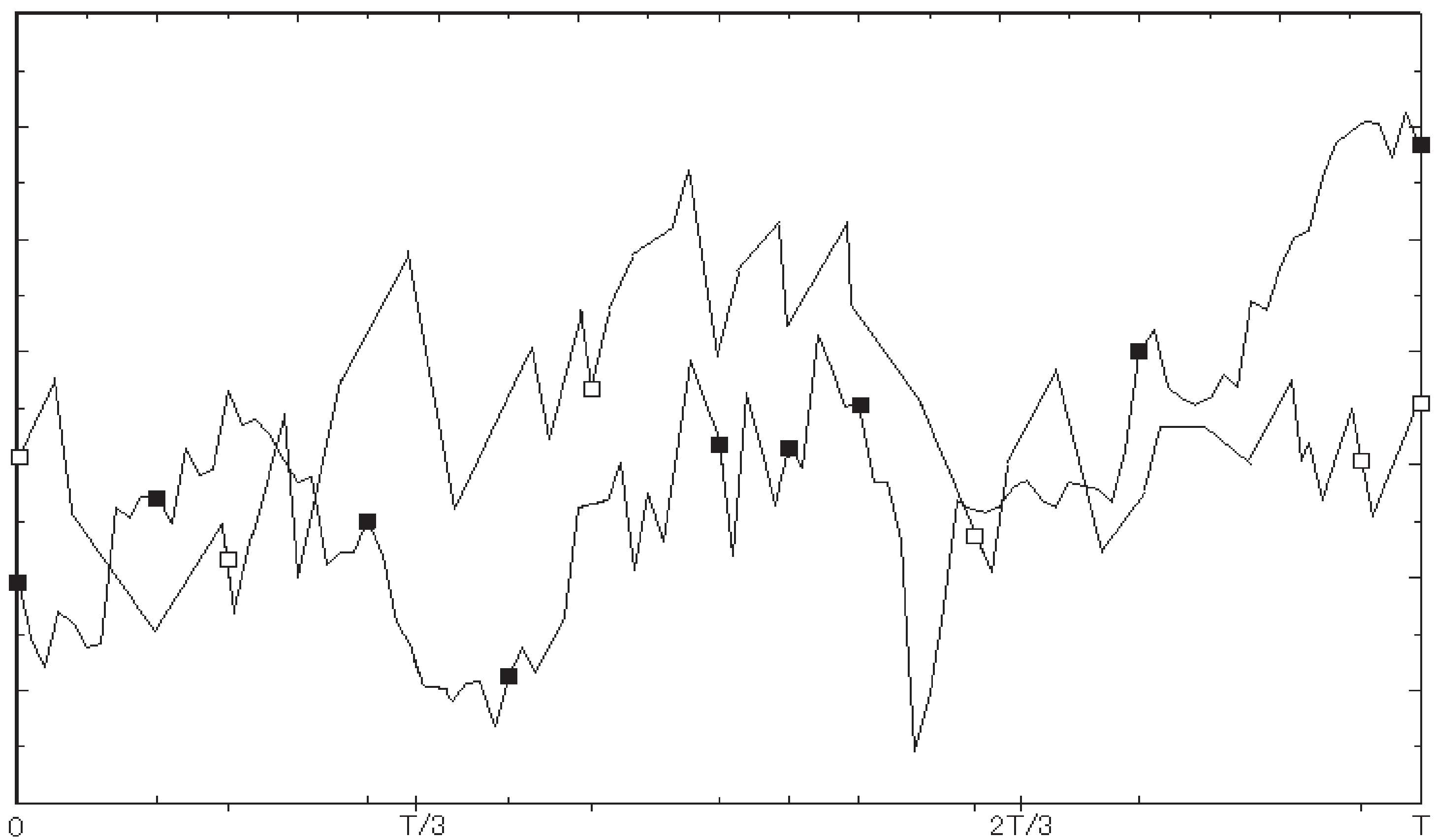
Irregularly observed points:

$$0 = t_0^i < t_1^i < \cdots < t_k^i < \cdots < t_{N_i-1}^i < t_{N_i}^i = T$$

Observed data:

$$p_i(t_0^i), p_i(t_1^i), \cdots, p_i(t_k^i), \cdots, p_i(t_{N_i-1}^i), p(t_{N_i}^i)$$

Nonsynchronous observations



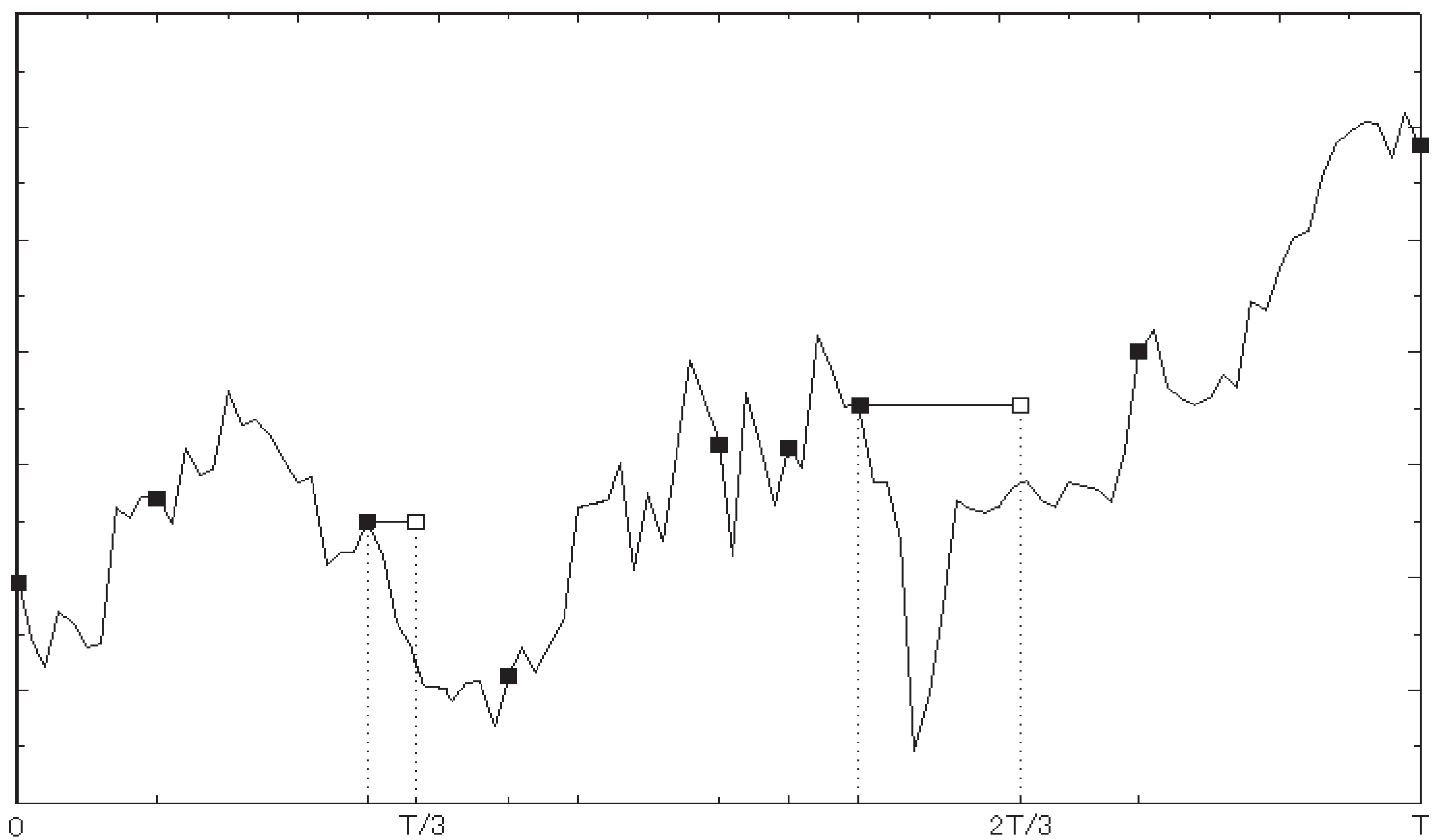
■ 1st asset; □ 2nd asset

- $$\overbrace{\{p_i(t_k^i)\}_{k=0}^{N_i}}^{\text{Unevenly spaced data}} \rightarrow \boxed{\text{Interpolation}} \rightarrow \overbrace{\{q_i(mT/M)\}_{m=0}^{M}}^{\text{Evenly spaced data}}$$

Previous-tick interpolation

$$q_i\left(\frac{mT}{M}\right) = p_i\left(\max\left(t_k^i : t_k^i \leq \frac{mT}{M}\right)\right)$$

Previous-tick interpolation



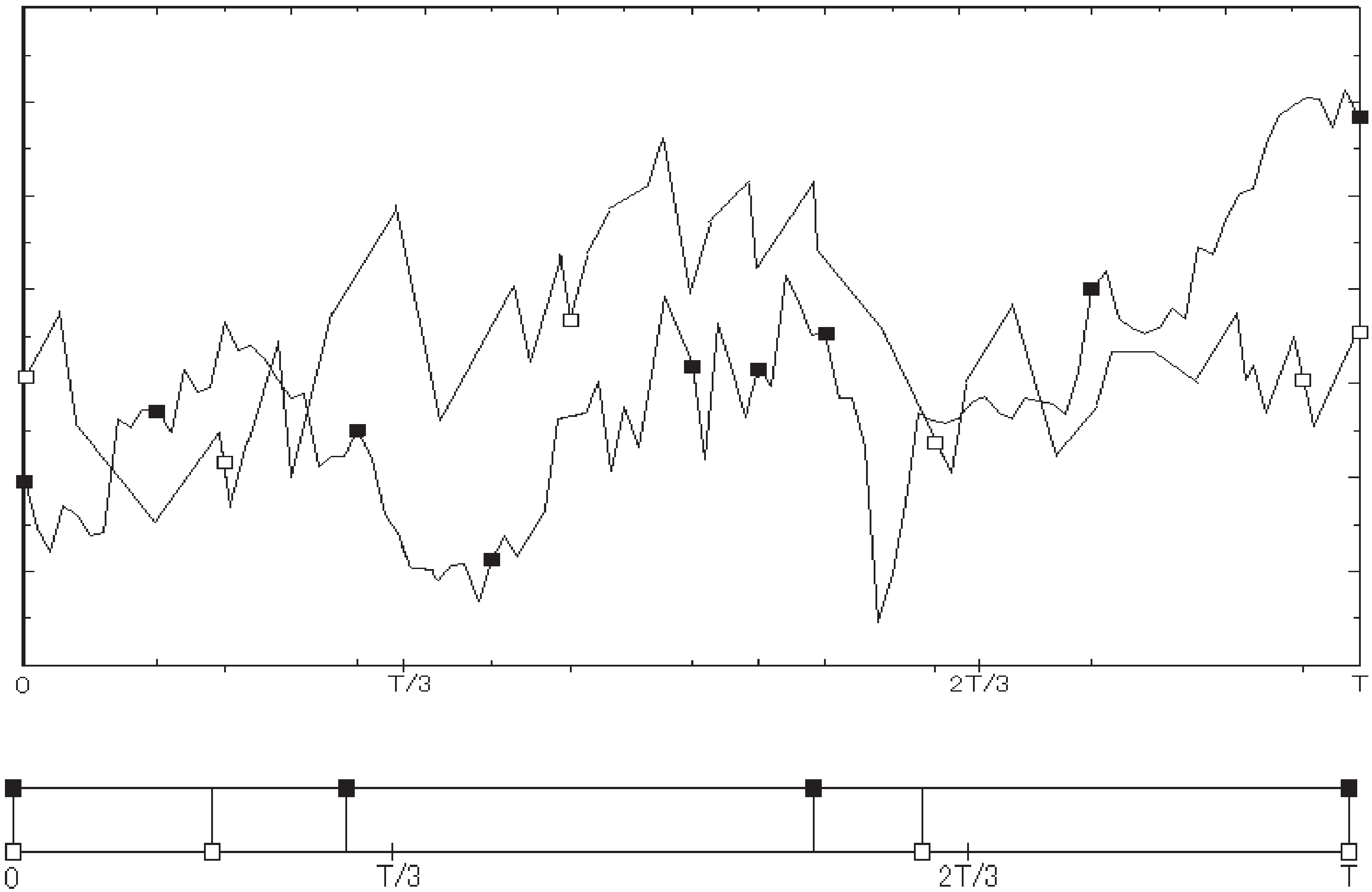
■ Raw data; □ Interpolated data

Realized covariance from the interpolated data

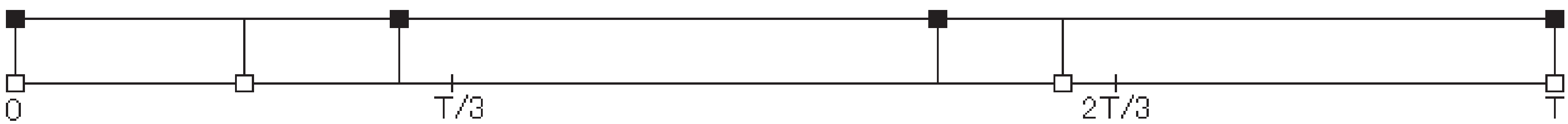
■ Realized covariance

$$\widehat{\omega}_{ij} = \sum_{m=1}^{M} \left\{ q_i\left(\frac{mT}{M}\right) - q_i\left(\frac{(m-1)T}{M}\right)\right\}\left\{ q_j\left(\frac{mT}{M}\right) - q_j\left(\frac{(m-1)T}{M}\right)\right\}$$

$\Rightarrow$ Interpolation bias $\Rightarrow$ Epps effect (Epps, 1979)


0
T/3
2T/3
T
0
T/3
2T/3
T

Interpolation bias



$$
\begin{aligned}
& E(\widehat{\omega}_{ij}) \\
= \quad & \sum_{m=1}^{M} E\left\{q_i\left(\frac{mT}{M}\right) - q_i\left(\frac{(m-1)T}{M}\right)\right\}\left\{q_j\left(\frac{mT}{M}\right) - q_j\left(\frac{(m-1)T}{M}\right)\right\} \\
= \quad & \sum_{m=1}^{M} \int_{I_m} \omega_{ij}(t)dt
\end{aligned}
$$

Bias-corrected estimator $\Rightarrow$

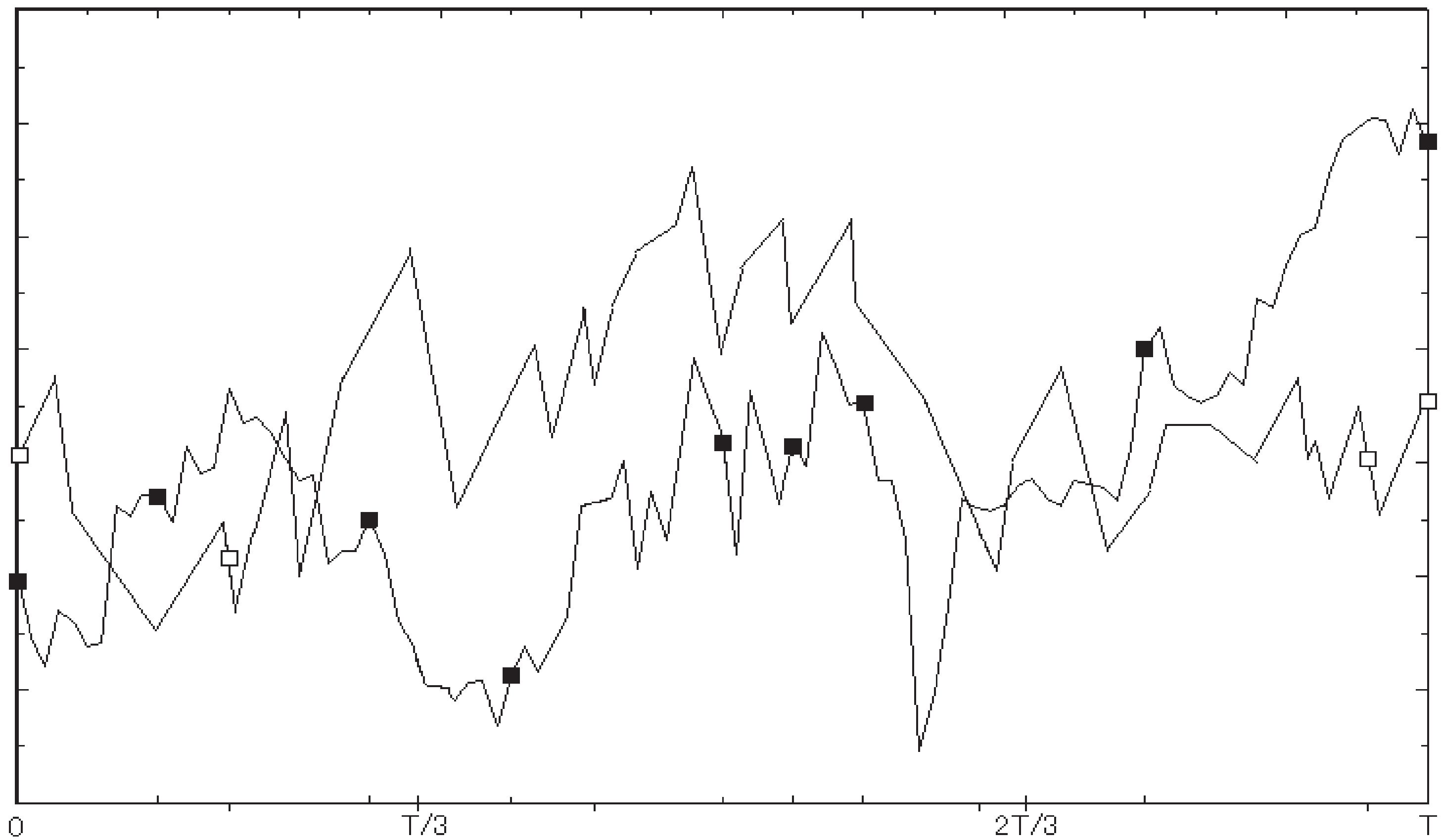
$$
\tilde{\omega}_{12} = \widehat{\omega}_{12} + \sum_{m=2}^{3} \Delta q_1(m-1)\Delta q_2(m) + \sum_{m=2}^{3} \Delta q_1(m)\Delta q_2(m-1)
$$

Bias-corrected realized covariance

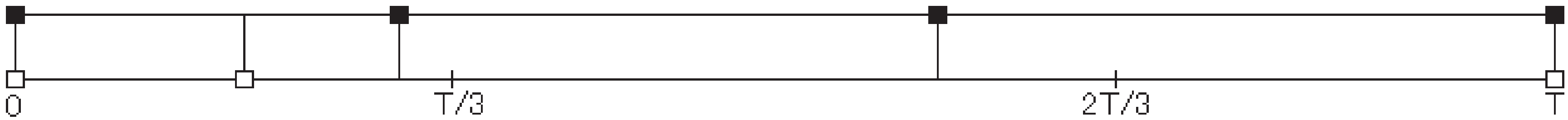
$$\tilde{\omega}_{12} = (\Delta q_1(1) \Delta q_1(2) \Delta q_1(3)) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

$$\tilde{\omega}_{ij} = \Delta q_i' \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \Delta q_j$$

More serious case



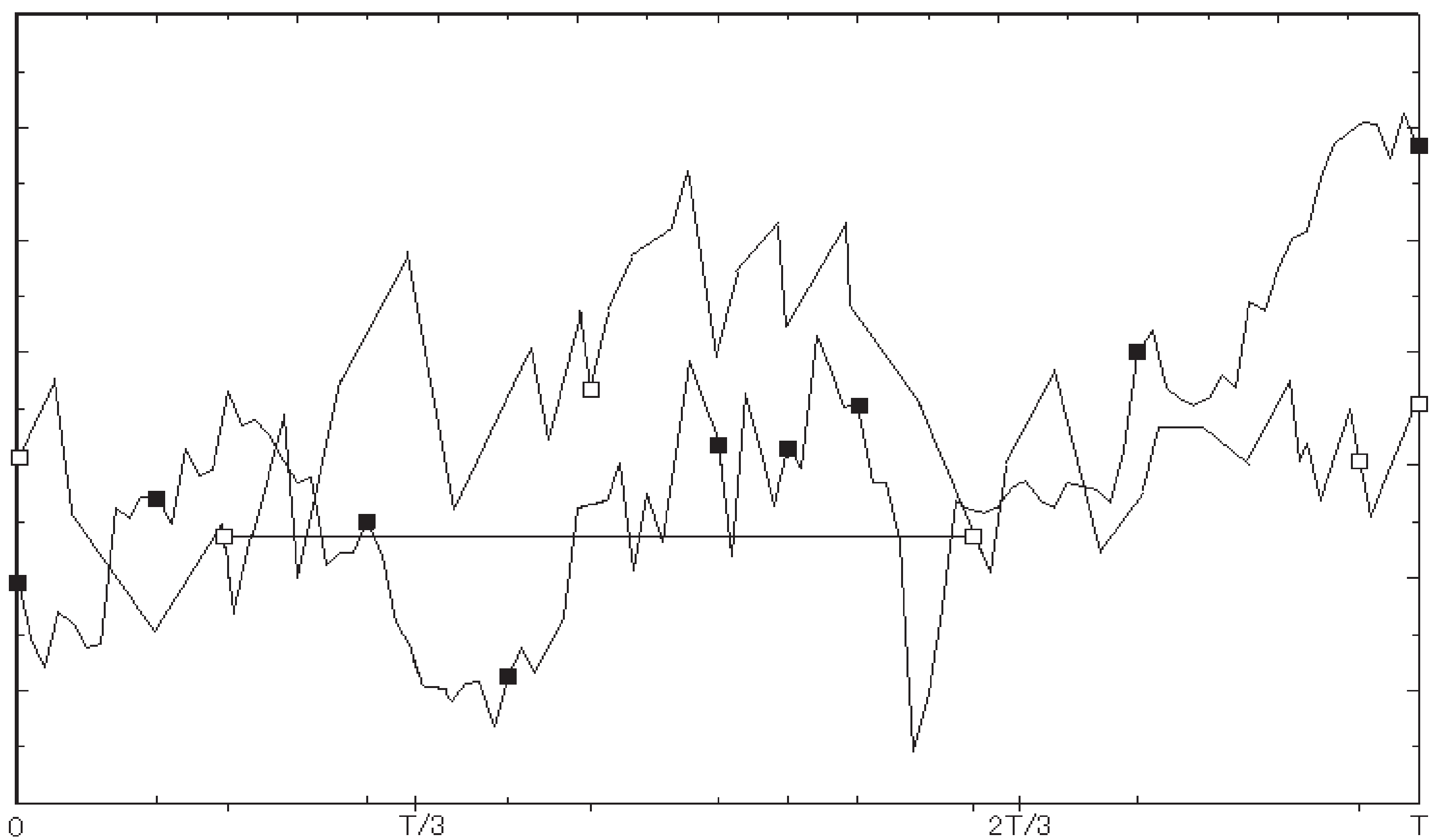
■ 1st asset; □ 2nd asset



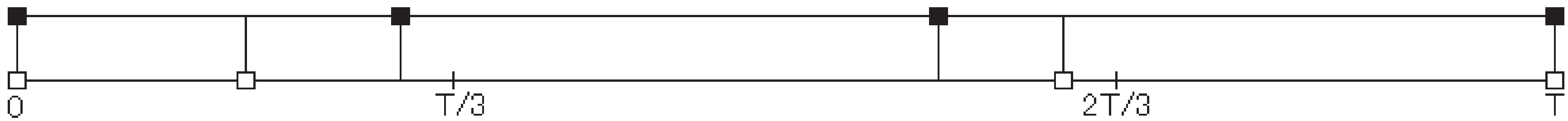
$$\tilde{\omega}_{12} = \hat{\omega}_{12} + \sum_{m=2}^{3} \Delta q_1(m-1)\Delta q_2(m) + \sum_{m=2}^{3} \Delta q_1(m)\Delta q_2(m-1)$$
$$+\Delta q_1(1)\Delta q_2(3)$$

$$= (\Delta q_1(1)\Delta q_1(2)\Delta q_1(3)) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

Another serious case



■ 1st asset; □ 2nd asset



$$\tilde{\omega}_{12} = \hat{\omega}_{12} + \sum_{m=2}^{3} \Delta q_1(m-1)\Delta q_2(m) + \sum_{m=2}^{3} \Delta q_1(m)\Delta q_2(m-1) + \Delta q_1(1)\Delta q_2(3)$$

$$= (\Delta q_1(1)\Delta q_1(2)\Delta q_1(3)) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

$$
\begin{aligned}
\tilde{\omega}_{ij} &= \Delta q_i' W \Delta q_j \\
&= \Delta q_i' \begin{pmatrix} 1 & 1 & 1_{1,3} & \cdots & 1_{1,M} \\ 1 & 1 & \ddots & \ddots & \vdots \\ 1_{3,1} & \ddots & \ddots & \ddots & 1_{M-2,M} \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1_{M,1} & \cdots & 1_{M,M-2} & 1 & 1 \end{pmatrix} \Delta q_j
\end{aligned}
$$

where

$$
W_{kl} = 1_{kl} = \begin{cases} 1 & \text{if } |k-l| \leq 1 \\ 1 & \text{if } k-l > 1 \text{ and } \Delta q_i(k-1) = \cdots = \Delta q_i(l+1) \\ 1 & \text{if } l-k > 1 \text{ and } \Delta q_j(l-1) = \cdots = \Delta q_j(k+1) \\ 0 & \text{otherwise} \end{cases}
$$

Monte Carlo study

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \ 0 \le t \le T$$

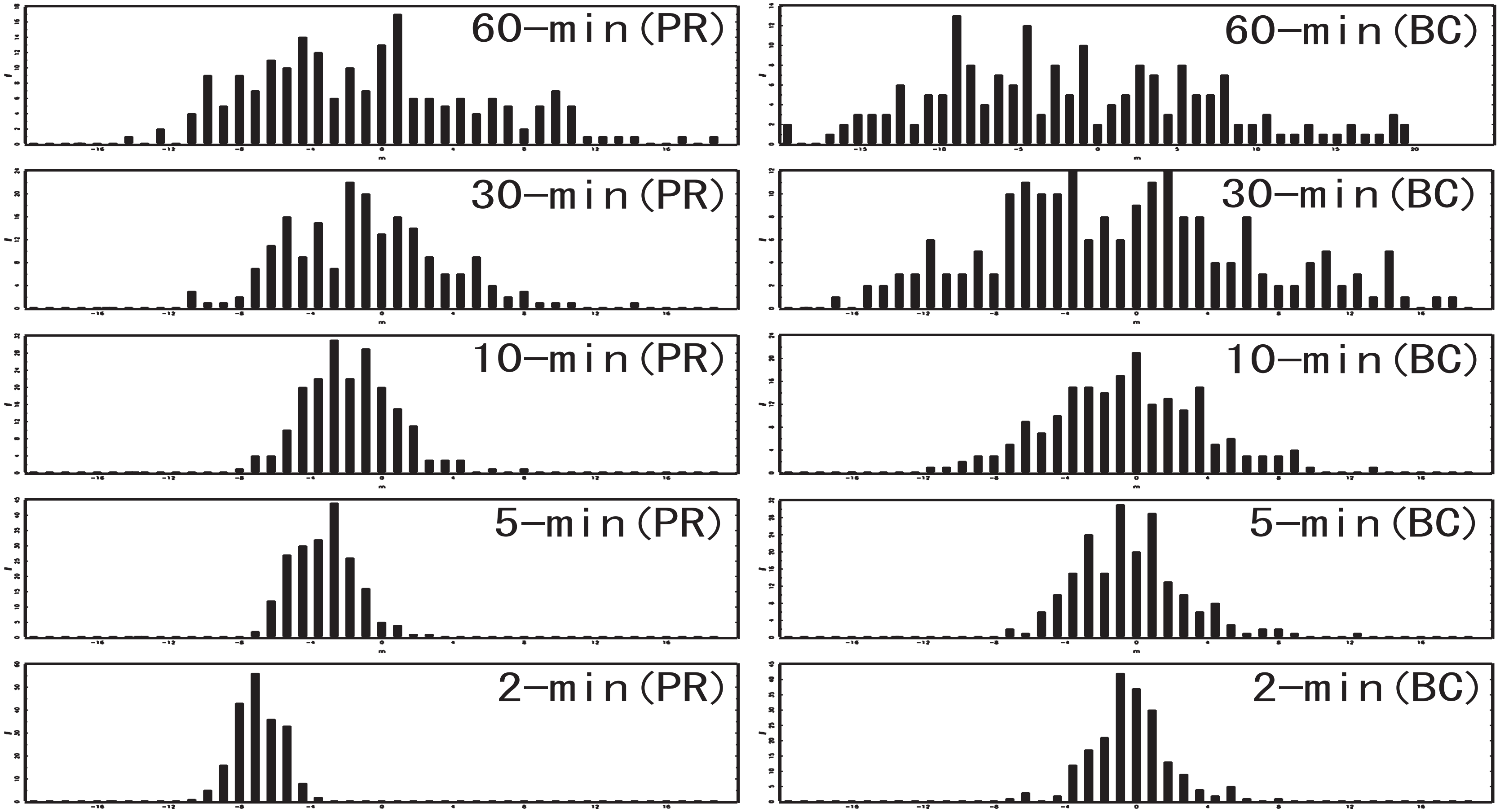
$$d\sigma_{ij}(t) = \kappa \left(\theta - \sigma_{ij}(t)\right) dt + \gamma dW_{ij}(t), \ i, j = 1, 2.$$

where $\kappa = 0.01$, $\theta = 0.01$, and $\gamma = 0.001$ and $T = 60 \times 60 \times 24$ seconds. Time differences are drawn from an exponential distribution with mean 45 seconds for p_1 and 60 seconds for p_2:

$$F\left(t_k^i - t_{k-1}^i\right) = 1 - \exp\left\{-\lambda_i \left(t_k^i - t_{k-1}^i\right)\right\}, \ i = 1, 2$$

where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_1 = 1/45$ and $\lambda_2 = 1/60$.

Distribution of measurement error



Summary

Contributions:

- Correction of interpolation bias

Remaining works:

- Evaluation of MSE
- Other microstructure biases

The full paper is available at `http://www.geocities.jp/kanatanit/`